ELSEVIER
Journal of Fluids and Structures 23 (2007) 1265–1270
JOURNAL OF FLUIDS AND STRUCTURES
www.elsevier.com/locate/jfs

Brief Communication

# On the use of negative penalty parameters in aeroelastic divergence analysis of lifting surfaces

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Received 25 November 2005; accepted 5 May 2007
Available online 20 September 2007

**Abstract**

This paper shows that a numerical modelling method in which constraints are replaced with positive and negative penalty functions, which may be regarded as artificial elastic restraints of positive and negative stiffness, may be safely used to determine the critical speed associated with aeroelastic divergence. The critical speeds of a beam with restraints of positive and negative stiffness are found to converge to that of the constrained system, from below if the stiffness is positive and from above otherwise. A uniform Euler–Bernoulli beam clamped at the rear end is analysed using an artificial restraint to enforce the constraint of zero rotation at the clamp, and the results are compared with the exact critical speed of the constrained system obtained analytically. The paper shows that, contrary to common belief that the penalty parameter must be positive, the inclusion of a negative penalty parameter enables the determination of errors due to violation of the constraints.


*Keywords:* Critical speed; Negative penalty parameter

## 1. Introduction

Aeroelastic divergence of lifting surfaces is a phenomenon in which the structure loses its stiffness due to destabilising aerodynamic forces. Except in some simple cases, exact analytical determination of critical speeds is not possible and the solution is commonly obtained by using numerical procedures such as the finite element method (Satt, 1992). In applying numerical methods, essential support and continuity conditions are often imposed approximately using the penalty function method. The main disadvantage of this method is the difficulty in choosing a suitable value for the penalty parameter which must be large enough to minimise any violation of the constraint condition, but must be small enough to avoid numerical problems such as ill-conditioning (Courant, 1943; Zienkiewicz, 1977). A penalty value is often chosen empirically or by trial and error until numerical convergence is observed.

The origin of the penalty method may be traced to the work by Courant (1943) in which he introduced the use of elastic restraints with high stiffness to represent rigid constraints thus relaxing the admissibility requirements. This method is now widely used in structural applications (Amabili and Garziera, 1999, 2000; Cheng and Nicolas, 1992; Courant, 1943; Gorman, 1989; Yuan and Dickinson, 1992). The same approach has also been adopted in the finite

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0889-9746/$ - see front matter 
doi:10.1016/j.jfluidstructs.2007.05.003

element method and other numerical procedures through the concept of penalty functions, in which quadratic error functions are multiplied by large penalty terms and included in the minimization equations to reduce the error due to violation of any constraints (Gavete et al., 2000; Pannachet and Askes, 2000; Zienkiewicz, 1977). Until recently the magnitude of the stiffness of the artificial restraint or the penalty parameter was selected either empirically or through a trial and error procedure until numerical convergence was observed, and the method did not provide any information on the maximum possible error due to violation of the constraints. However, recent publications show that by using a combination of positive and negative stiffness values (or penalty values), it is possible to delimit the natural frequencies and critical loads of structures (Ilanko, 2002a, 2003; Ilanko and Dickinson, 1999). Some numerical results show that bounding and converging values for the deflection of a constrained structure may be obtained from asymptotic models with positive and negative penalty functions (Ilanko, 2002b) and a proof of convergence for this case has recently been published (Ilanko, 2005). Similar findings have also been confirmed for a simple boundary value problem using a meshless discretisation method (Askes and Pannachet, 2005) and in the solution of a heat transfer problem (Ilanko and Tucker, 2005). More recently it has been shown that plotting the variation of a required parameter against the inverse of penalty parameters is more informative and useful (Askes and Ilanko, 2006; Williams and Ilanko, 2006).

The purpose of this note is to show that such an asymptotic modelling method with positive and negative penalty terms also works well in the determination of critical speed associated with aeroelastic divergence. An analytical formulation for the aerodynamic force acting on a low aspect ratio wing bending in the chordwise direction, based on the linearised supersonic wing theory was presented by Bisplinghoff et al. (1955). Based on this theory, the divergence analysis of a lifting surface, namely a cantilever beam, has been carried out using Galerkin's method with displacement functions, which individually violate the zero slope condition. The zero slope condition has then been enforced approximately by applying a penalty against the violation of this constraint, using large positive and negative penalty parameters. The results compare well with exact results (Satt, 1992).

## 2. System description

Consider the system shown in Fig. 1, a uniform beam of length $L$, flexural rigidity $EI$ clamped at the rear end (with respect to the direction of flight) as an illustrative example. The beam is subject to an aerodynamic force of intensity (local lift per unit length)

$$f = f' \frac{du}{dx}, \tag{1}$$

where $f'$ is the lift slope per unit length and is given by

$$f' = \frac{4M^2}{\sqrt{M^2 - 1}} \frac{\rho_\infty a_\infty^2 b}{2}. \tag{2}$$

Here, $\rho_\infty$ is the density of air, $a_\infty$ is the speed of sound, $b$ is the width of the beam, $du/dx$ is the local slope of the transverse displacement $u$ and $M$ is the Mach number given by the ratio of airspeed to the speed of sound $a_\infty (M = v/a_\infty)$.

The governing equation for the system is

$$EI \frac{\partial u^4}{\partial x^4} - f' \frac{\partial u}{\partial x} = 0. \tag{3}$$

The exact solution to the problem is available and will be used for verifying the accuracy of the numerical procedure. Since the purpose of this note is to show the applicability of the negative penalty method, one of the constraints will be replaced with an elastic restraint. Relaxing the zero slope condition at the clamp and introducing an artificial rotational elastic restraint with a stiffness $k_\theta$ at the rear end of the beam (at $x = 0$), the model in Fig. 2 is derived. The only geometric boundary condition for this system is that the translation at the support is zero, $u(0) = 0$.

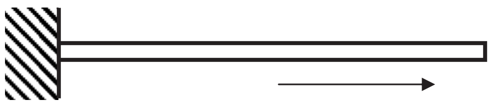


Fig. 1. Constrained system (clamped beam).

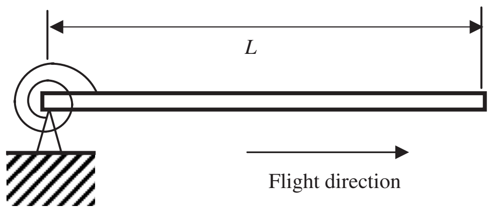


Fig. 2. Approximate model.

## 3. Solution by the Galerkin method

The displacement $u$ may be expressed as a series of admissible functions:

$$u = \sum_{i=1}^{n} c_i \phi_i(x), \tag{4}$$

where the functions must satisfy the constraint condition

$$\phi_i(0) = 0. \tag{5}$$

Eq. (3), the governing differential equation for the system in Fig. 1, also governs the model in Fig. 2. Substituting Eq. (4) into Eq. (3) gives

$$EI \sum_{i=1}^{n} c_i \phi_i'''' - f' \sum_{i=1}^{n} c_i \phi_i' = 0. \tag{6}$$

The Galerkin method formulation leads to $n$ number of equations of which the $j$th equation is

$$\int_{x=0}^{L} \left( EI\phi_j \sum_{i=1}^{n} c_i \phi_i'''' - f'\phi_j \sum_{i=1}^{n} c_i \phi_i' \right) \mathrm{d}x = 0. \tag{7}$$

The first integral may be transformed into a weak form by integration by parts giving

$$\int_{x=0}^{L} EI\phi_j \sum_{i=1}^{n} c_i \phi_i'''' \,\mathrm{d}x = \int_{x=0}^{L} EI \sum_{i=1}^{n} c_i \phi_i'' \phi_j'' \,\mathrm{d}x + EI \sum_{i}^{n} c_i \phi_j \phi_i''' \Big|_0^L - EI \sum_{i}^{n} c_i \phi_j' \phi_i'' \Big|_0^L. \tag{8}$$

For the approximate model, all boundary values vanish except $EI\sum_i^n c_i \phi_j' \phi_i''\big|_0$. This term represents the energy stored in the artificial rotational restraint. Since $EI\sum_i^n c_i \phi_i''\big|_0$ represents the moment at $x = 0$, we can write $EI\sum_i^n c_i \phi_i''\big|_0 = \sum_i^n k_\theta c_i \phi_i'(0)$.

Therefore, the weak form of the equation is

$$\int_{x=0}^{L} \left( EI \sum_{i=1}^{n} c_i \phi_i'' \phi_j'' - f'\phi_j \sum_{i=1}^{n} c_i \phi_i' \right) \mathrm{d}x + \sum_{i=1}^{n} k_\theta c_i \phi_i'(0) \phi_j'(0) = 0. \tag{9}$$

This may be written in matrix form as

$$[G]\{c\} - f'[F]\{c\} = \{0\}, \tag{10}$$

where

$$G_{i,j} = \int_{x=0}^{L} EI\phi_i'' \phi_j'' \,\mathrm{d}x + k_\theta \phi_i'(0)\phi_j'(0) \quad \text{and} \quad F_{i,j} = \int_{x=0}^{L} \phi_j \phi_i' \,\mathrm{d}x. \tag{10a,b}$$

The shape functions are taken as simple polynomials of the form

$$\phi_i = x^i. \tag{11}$$

This yields the following expressions for the matrix elements:

$$G_{i,j} = k_\theta \quad \text{if } i = j = 1,$$

$$G_{i,j} = 0 \quad \text{if } i = 1 \text{ or } j = 1 \text{ but } i \neq j,$$

$$G_{i,j} = \frac{EIi(i-1)j(j-1)L^{(i+j-3)}}{(i+j-3)} \quad \text{if } i \neq 1 \text{ and } j \neq 1 \tag{12}$$

and

$$F_{i,j} = \frac{iL^{(i+j)}}{(i+j)}. \tag{13}$$

Pre-multiplying Eq. (10) by $[G]^{-1}$ leads to the eigenvalue problem

$$[G]^{-1}[F]\{c\} = 1/f'[I]\{c\}. \tag{14}$$

For a non-trivial solution, $1/f'$ must be an eigenvalue of $[G]^{-1}[F]$. The lowest critical velocity is related to the largest eigenvalue $\lambda$. Results of the largest eigenvalue as a function of $k_\theta$ is given in Fig. 3 for the following data: $L = 0.12$ m, $EI = 3.77\,\text{N m}^2$, $\rho_\infty = 1.225\,\text{kg/m}^3$, $a_\infty = 340.3$ m/s, $b = 0.01$ m.

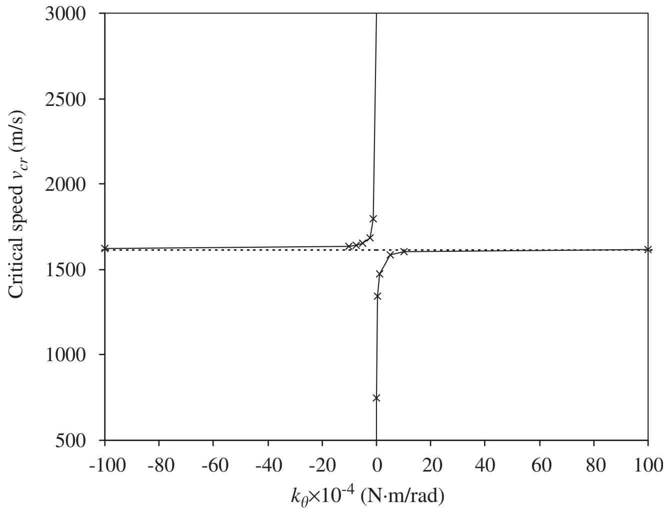


Fig. 3. Variation of the critical speed with stiffness of the restraint: ⋯⋯, constrained system; —×—, asymptotic model.

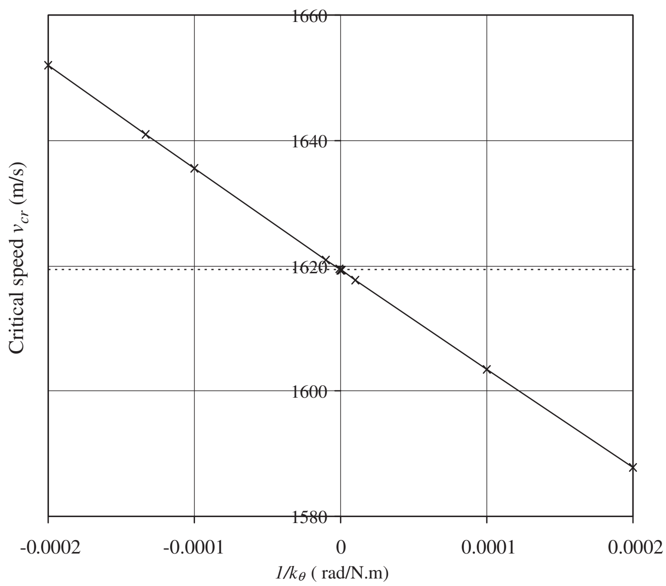


Fig. 4. Variation of the critical speed with inverse stiffness: ⋯⋯, constrained system; —×—, asymptotic model.

Table 1
Numerical results

| $k_\theta$ (N m/rad) | $\lambda_{\max}$ (m/N) | $v_{cr}$ (m/s) |
|---|---|---|
| $10^9$ | 0.000072413 | 1619.4 |
| $10^8$ | 0.000072413 | 1619.4 |
| $10^7$ | 0.000072414 | 1619.4 |
| $10^6$ | 0.000072420 | 1619.2 |
| $10^5$ | 0.000072482 | 1617.8 |
| $10^4$ | 0.000073096 | 1603.5 |
| $10^3$ | 0.000079272 | 1472.1 |
| $10^2$ | 0.000142536 | 749.8 |
| $-10^2$ | 0.000017567 | 6819.1 |
| $-10^3$ | 0.000065629 | 1794.4 |
| $-10^4$ | 0.000071731 | 1635.5 |
| $-10^5$ | 0.000072345 | 1621.0 |
| $-10^6$ | 0.000072407 | 1619.5 |
| $-10^7$ | 0.000072413 | 1619.4 |
| $-10^8$ | 0.000072413 | 1619.4 |
| $-10^9$ | 0.000072413 | 1619.4 |
| $\pm\infty$ (constrained) | 0.000072413 | 1619.4 |

The critical velocity in terms of the highest eigenvalue $\lambda_{\max}$ is

$$v_{cr} = \sqrt{\frac{\left((a_\infty^{-2}) \pm \sqrt{(a_\infty^{-4} - 16\lambda_{\max}^2\rho_\infty^2 b^2}\right)}{8\lambda_{\max}^2\rho_\infty^2 b^2}}. \tag{15}$$

The above equation gives two possible solutions due to the plus or minus term in the numerator. However, using the minus sign gives a result very close to the speed of sound where Eq. (2) is not applicable. Therefore, only the solution associated with the plus sign is used. The results for the critical velocity for various values of positive and negative stiffness are presented graphically in Fig. 4. Some numerical results are given in Table 1. Six terms were ($n = 6$) found to be sufficient to obtain the eigenvalues presented here but convergence was checked by taking more terms up to $n = 10$, and computing the results to nine decimal places. The critical speed for the cantilever beam calculated for the same data using the exact method described in Satt, (1992) is 1619.4 m/s. The results for positive and negative values of stiffness bound the exact results for the fully constrained system. Interestingly, plotting the critical speed against the inverse stiffness parameter results in a near-straight line for moderately large stiffness terms, in line with a similar phenomenon observed in the solution of other constrained variational problems (Askes and Ilanko, 2006; Williams and Ilanko, 2006). Fig. 4 shows such a plot for $k_\theta > 5000$ N m/rad.

It should be stated here that the use of positive and negative penalty parameters helps to calculate and control the maximum possible error due to violation of the constraints only. While this removes some limitations on the choice of admissible functions, which may improve the accuracy of the overall solution, the actual error due to Galerkin's approximation cannot be estimated using this method.

## 4. Concluding remarks

The critical speed of air due to aerodynamic forces acting on a beam partially restrained by an artificial elastic restraint was determined using the Galerkin method. The critical speed thus calculated approaches that the fully constrained beam as the magnitude of the stiffness becomes very large but its direction of approach depends on the sign of the stiffness parameter used. This shows that any error in the calculation of critical speeds due to approximation of constraints with artificial restraints or penalty parameters can be determined and controlled.

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